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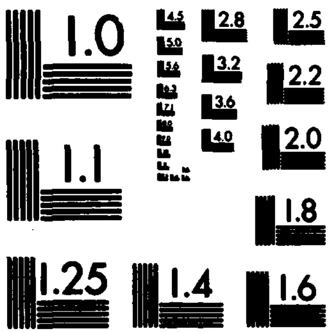

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Dennis A. Brindley
Bristol-Myers Pharmaceutical Research and Development Division
and
Ralph A. Bradley
University of Georgia and Florida State University

The Florida State University

Department of Statistics

Tallahassee, Florida
32306

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The University of Georgia
Department of Statistics and Computer Science
Athens, Georgia 30602

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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SOME NEW RESULTS ON GRUBBS' ESTIMATORS

DENNIS A. BRINDLEY AND RALPH A. BRADLEY*

Consider a two-way classification with n rows and r columns and the usual model of analysis of variance except that the error components of the model may have heterogeneous variances $\cancel{\sigma_j^2, j = 1, \ldots, r}$ by columns. Grubbs provided unbiased estimators Q_j of σ_j^2 that depend on column sums of squared residuals S_j. When $r = 3$, the joint distributions of the S_j and the Q_j are given for the first time in closed form.

Two tests proposed by Russell and Bradley are examined when $r = 3$, one for variance homogeneity and the second for one possible disparate variance. A very simple distribution is found for the test statistic of the first test and its non-null distribution is derived also. The distribution of the second test statistic was known to be the central variance-ratio distribution in the null case and now its ratio to a parameter of noncentrality is shown to have that same distribution in the non-null case.

When $r = 4$, $n = 4$, the joint distribution of the S_j is given also in closed form, but it is difficult to use. For $r > 3$, an approximate test of variance homogeneity is proposed, based on an extension of the Russell-Bradley statistic. Extensive simulation studies show that the distribution of the test statistic may be approximated very well by a chi-square distribution.

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* Dennis A. Brindley is a research scientist with Bristol-Myers Pharmaceutical Research and Development Division, Evansville, Indiana 47630. Ralph A. Bradley is Research Professor, Department of Statistics and Computer Science, University of Georgia, Athens, Georgia 30602.

1. INTRODUCTION

Consider the two-way classification of observations y_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, r$, and the model,

$$y_{ij} = \mu_i + \beta_j + \varepsilon_{ij}, \tag{1}$$

when μ_i represents the mean response of row i, β_j represents the additional effect of column j, $\sum_j \beta_j = 0$, and the ε_{ij} are independent, zero-mean, normal variates with expectation $E(\varepsilon_{ij}^2) = \sigma_j^2$. Model (1) differs from the usual model in that variances are column-related. Grubbs (1948) estimated σ_j^2 using Q_j, $j = 1, \ldots, r$, where

$$Q_j = \frac{r}{(n-1)(r-2)} S_j - \frac{1}{(n-1)(r-1)(r-2)} \sum_{t=1}^{r} S_t \tag{2}$$

and

$$S_j = \sum_{i=1}^{n} (y_{ij} - y_{i.} - y_{.j} + y_{..})^2 \tag{3}$$

in which $y_{i.}$, $y_{.j}$ and $y_{..}$ are respectively the means of the observations in the i-th row, j-th column, and complete table. Note that $E(Q_j) = \sigma_j^2$, but Q_j may be negative in certain situations.

Possible use of Grubbs' estimators, and inferences based upon them, arises frequently. Grubbs was concerned with the variabilities of observer-used electric clocks as timing instruments, Ehrenberg (1950) with the precisions of individuals in the sensory scoring of food samples, and Russell and Bradley (1958) with an application to three

fermentors in a distillery. More recently, Snee (1982) examined wheat-yield data and multiple-head machine data and suggested that apparent variance heterogeneity may be an indicator and manifestation of omitted interaction terms needed in the model. A substantial bibliography on Grubbs' estimators has developed and was summarized by Maloney (1973).

The use of Grubbs' estimators has been hampered by the difficulties encountered in developing appropriate inference procedures, although some results are available. Russell and Bradley gave two procedures. The first was based on the assumption,

$$A: \quad \sigma_j^2 = \sigma^2, \; j = 1, \ldots, (r - 1), \tag{4}$$

and provided a test of the hypothesis,

$$H_{01}: \quad \sigma_r^2 = \sigma^2. \tag{5}$$

They showed that

$$F = [r(r - 2)Q_r + \sum_j Q_j]/r(\sum_j Q_j - Q_r) \tag{6}$$

has the central F-distribution with $(n - 1)$ and $(n - 1)(r - 2)$ degrees of freedom under A and H_{01}. In the special case with $r = 3$, they considered the hypothesis,

$$H_{02}: \quad \sigma_1^2 = \sigma_2^2 = \sigma_3^2, \tag{7}$$

and showed that, under H_{02}, the statistic,

$$S = -(n - 1)\log T, \tag{8}$$

where

$$T = 3(Q_1Q_2 + Q_1Q_3 + Q_2Q_3)/(Q_1 + Q_2 + Q_3)^2, \tag{9}$$

has the central chi-square distribution with 2 degrees of freedom asymptotically with n. Shukla (1982) proposed a Bartlett-type statistic, which, in the notation of this paper, depends on log T* where $T^* = \prod_j S_j/\bar{S}^r$ with $\bar{S} = \sum_j S_j/r$. He also approximated the distribution of log T*. Johnson (1962) specified j and j', j ≠ j', and used $S_j/S_{j'}$ to test the hypothesis that $\sigma_j^2 = \sigma_{j'}^2$, obtaining a distribution that he approximated with a single F-distribution.

Ellenberg (1977), under the hypothesis that $\sigma_j^2 = \sigma^2$, $j = 1, \ldots, r$, derived the exact distribution of a subset, $S_1, \ldots, S_k$, $k \leq r$, of $S_1, \ldots, S_r$. A complicated infinite series expression resulted that he regarded as intractable for further use.

In this article, a direct attempt is made to obtain the joint distribution of $S_1, \ldots, S_r$ and of $Q_1, \ldots, Q_r$ that is partially successful. Simple results are obtained when r = 3, the most important case. Then the exact distribution of T in (9) is derived both under H_{01} in (7) and under the general alternative hypothesis, H_{02}: $\sigma_j^2 \neq \sigma^2$ for some j = 1, 2, 3. The non-null distribution of F in (6) is found also. When r = 4, the joint distribution of $S_1, \ldots, S_4$ is obtained under the extension of (7) for r = 4, but this distribution is complicated, although given in closed form. In general, when r > 3, distributional problems become very difficult; an approximate test procedure for equality of variances is then suggested.

2. DISTRIBUTION THEORY

We seek the joint distribution of $S_1, \ldots, S_r$, $r \geq 3$. Let $\underset{\sim}{y}$ be the column vector with typical element y_{ij}, the elements arranged in lexicographic order. Define $\underset{\sim}{A}_n$ to be the $(n - 1) \times n$ matrix with the $(n - 1)$ zero-sum rows of the n-dimensional Helmert matrix arranged so that row s has s elements $1/[s(s + 1)]^{\frac{1}{2}}$ followed by the element $-s/[s(s + 1)]^{\frac{1}{2}}$ and $(n - s - 1)$ zero elements. The usual $(n - 1)(r - 1)$ error contrasts may be written as the elements of

$$\underset{\sim}{e} = (\underset{\sim}{A}_n \otimes \underset{\sim}{A}_r)\underset{\sim}{y}, \tag{10}$$

where the typical element of $\underset{\sim}{e}$ is e_{ij}, $i = 1, \ldots, (n - 1)$, $j = 1, \ldots, (r - 1)$, arranged in lexicographic order and $\underset{\sim}{A}_n \otimes \underset{\sim}{A}_r$ is the direct product of $\underset{\sim}{A}_n$ and $\underset{\sim}{A}_r$.

It is clear that $E(\underset{\sim}{e}) = \underset{\sim}{0}$ and it may be shown that the covariance matrix of $\underset{\sim}{e}$ is $\underset{\sim}{\Sigma}_e = \underset{\sim}{I}_{(n-1)} \otimes \underset{\sim}{\Sigma}$, where $\underset{\sim}{I}_m$ is the m-dimensional identity matrix. The matrix $\underset{\sim}{\Sigma}$ is $(r - 1)$-square symmetric with diagonal elements,

$\left(\sum_{j=1}^{s} \sigma_j^2 + s^2\sigma_{s+1}^2\right)/s(s + 1)$, and (s, t)-elements,

$\left(\sum_{j=1}^{t} \sigma_j^2 - t\sigma_{t+1}^2\right)/[t(t + 1)s(s + 1)]^{\frac{1}{2}}$, $t < s$, $s = 1, \ldots, (r - 1)$.

The form of $\underset{\sim}{\Sigma}_e$ indicates that $\underset{\sim}{e}$ may be partitioned into $(n - 1)$ independent $(r - 1)$-dimensional vectors, each having covariance matrix $\underset{\sim}{\Sigma}$, $\underset{\sim}{e}' = (\underset{\sim}{e}_1', \ldots, \underset{\sim}{e}_{n-1}')$. In effect, $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{n-1}$ may be regarded as $(n - 1)$ independent observation vectors from an $(r - 1)$-dimensional multivariate normal population with mean vector $\underset{\sim}{0}$ and dispersion matrix $\underset{\sim}{\Sigma}$.

Let $\underset{\sim}{V} = \sum_{i=1}^{n-1} \underset{\sim}{e}_i \underset{\sim}{e}_i'$. Then the $r(r - 1)/2$ distinct elements of V have the Wishart density with $(n - 1)$ degrees of freedom and dispersion

matrix $\underset{\sim}{\Sigma}$ so that

$$f(\underset{\sim}{V};\underset{\sim}{\Sigma}) = \frac{|\underset{\sim}{V}|^{(n-r-1)/2}\exp(-\frac{1}{2}\,\mathrm{tr}\underset{\sim}{V}\underset{\sim}{\Sigma}^{-1})}{2^{(n-1)(r-1)/2}\pi^{(r-1)(r-2)/4}|\underset{\sim}{\Sigma}|^{(n-1)/2}\prod_{j=1}^{r-1}\Gamma[\frac{1}{2}(n-j)]} \tag{11}$$

for $\underset{\sim}{V}$ positive definite and zero otherwise, f being used generically to denote a density function. Furthermore, if $v_{jj'}$ is the typical element of $\underset{\sim}{V}$, algebraic reduction of (3) yields

$$S_j = [(j-1)/j]v_{j-1,j-1} + \sum_{t=j}^{r-1}[v_{tt}/t(t+1)] + \tag{12}$$

$$2\left[\sum_{j\le t<t'<r} v_{tt'}/\{t(t+1)t'(t'+1)\}^{\frac{1}{2}} - \{(j-1)/j\}^{\frac{1}{2}}\sum_{t=j}^{r-1} v_{j-1,t}/\{t(t+1)\}^{\frac{1}{2}}\right],$$

$j = 1, \ldots, r.$

When $\sigma_j^2 = \sigma^2$, $j = 1, \ldots, r$, $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{I}_{r-1}$ and $\underset{\sim}{\Sigma}^{-1} = \underset{\sim}{I}_{r-1}/\sigma^2$. The statistics F, S, and T in (6), (8) and (9) respectively, and any extensions to be considered, are scale invariant and we may take $\sigma^2 = 1$ without loss of generality when considering the distributions of such statistics when $\sigma_j^2 = \sigma^2$, $j = 1, \ldots, r$. Then

$$f(\underset{\sim}{V};\underset{\sim}{I}) = \frac{|\underset{\sim}{V}|^{(n-r-1)/2}\exp(-\frac{1}{2}\,\mathrm{tr}\underset{\sim}{V})}{2^{(n-1)(r-1)/2}\,\pi^{(r-1)(r-2)/4}\prod_{j=1}^{r-1}\Gamma[\frac{1}{2}(n-j)]} \tag{13}$$

for $\underset{\sim}{V}$ positive definite and zero otherwise.

In this section, we have reduced the apparent dependence of the S_j, and the Q_j, on the nr original observations to dependence on $\frac{1}{2}r(r-1)$ new variables with known joint distribution (11). It is clear from (12) that only a non-singular linear transformation is

needed to obtain the joint distribution of S_1, S_2 and S_3 when $r = 3$. We turn to special cases.

3. SPECIAL CASES

3.1 Joint Distributions, $r = 3$: When $r = 3$, the joint distributions of S_1, S_2 and S_3 and of Q_1, Q_2 and Q_3 follow directly from (11), (12) and (2). Now

$$\begin{aligned} S_1 &= (v_{11}/2) + (v_{22}/6) + (v_{12}/\sqrt{3}), \\ S_2 &= (v_{11}/2) + (v_{22}/6) - (v_{12}/\sqrt{3}), \qquad (14) \\ S_3 &= 2v_{22}/3, \end{aligned}$$

$$\underset{\sim}{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\Sigma} = \begin{bmatrix} (\sigma_1^2 + \sigma_2^2)/2 & (\sigma_1^2 - \sigma_2^2)/\sqrt{12} \\ (\sigma_1^2 - \sigma_2^2)/\sqrt{12} & (\sigma_1^2 + \sigma_2^2 + 4\sigma_3^2)/6 \end{bmatrix}.$$

The linear transformation (14) lets us write

$$f(S_1,S_2,S_3;\underset{\sim}{\Sigma}) = \frac{1}{\pi\Gamma(n-2)}\left(\frac{9}{4\sum_{j<j'}\sigma_j^2\sigma_{j'}^2}\right)^{\frac{1}{2}(n-1)}\left[(\sum_j S_j)^2 - 2\sum_j S_j^2\right]^{\frac{1}{2}(n-4)}$$
$$\exp\left[-\frac{1}{\sum_{j<j'}\sigma_j^2\sigma_{j'}^2}\left(\sum_{j\neq j'}\sigma_j^2 S_{j'} - \frac{1}{2}\sum_j \sigma_j^2 S_j\right)\right], \qquad (15)$$

where $0 < S_j < 2\sum_{j'} S_{j'}/3$, $j = 1, 2, 3$, and $(\sum_j S_j)^2 - 2\sum_j S_j^2 > 0$. From (2), the density of Q_1, Q_2 and Q_3 is

$$f(Q_1, Q_2, Q_3; \underset{\sim}{\Sigma}) = \frac{(n-1)^{n-1}}{4\pi\Gamma(n-2)(\sum_{j<j'}\sigma_j^2\sigma_{j'}^2)^{\frac{1}{2}(n-1)}}(Q_1Q_2 + Q_1Q_3 + Q_2Q_3)^{\frac{1}{2}(n-4)}$$
$$\exp\left(-\frac{n-1}{2\sum_{j<j'}\sigma_j^2\sigma_{j'}^2}\sum_{j\neq j'}\sigma_j^2 Q_{j'}\right), \qquad (16)$$

where $Q_1 + Q_2 + Q_3 > 0$ and $Q_1Q_2 + Q_1Q_3 + Q_2Q_3 > 0$. Note that one Q_j may be negative but not two.

3.2 Russell-Bradley Test, $r = 3$: The non-null distribution of F in (6) may be found from (16) when $\sigma_1^2 = \sigma_2^2 = \sigma^2$, $\sigma_3^2 \neq \sigma^2$. Now $F = [1 + 4Q_3/(Q_1 + Q_2)]/3$ and its density is

$$f(F;\gamma) = \gamma^{\frac{1}{2}(n-1)}F^{\frac{1}{2}(n-3)}(\gamma + F)^{-(n-1)}/B[\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1)], \quad (17)$$

where $F > 0$, $\gamma = (\sigma^2 + 2\sigma_3^2)/3\sigma^2$. Thus, for this test, F/γ has the central F-distribution with $(n - 1)$ and $(n - 1)$ degrees of freedom under both H_{01} in (5), for which $\gamma = 1$, and $H_{a1}:\sigma_3^2 \neq \sigma^2$ when $r = 3$.

Details on this result are given by Brindley (1982). The transformation, $x = Q_1 + Q_2$, $y = Q_1/(Q_1 + Q_2)$, $z = Q_3/(Q_1 + Q_2)$, is used to obtain (17) from (16) since the marginal distribution of z may be obtained and F is directly dependent on z.

To illustrate the results of this subsection, suppose that a one-sided alternative hypothesis is considered, say $H_{a1}:\sigma_3^2 > \sigma^2$. Let F^* be a random variable with the central F-distribution with $(n - 1)$ and $(n - 1)$ degrees of freedom and let F_α be such that $P(F^* > F_\alpha) = \alpha$. If F is computed from (6) with $r = 3$, H_{01} in (5), given A in (4), is rejected at significance level α if $F > F_\alpha$. The power of the test depends on γ, specified when σ_3^2/σ^2 is specified. The power of the test for specified γ is evaluated simply as $P(F^* > F_\alpha/\gamma)$. The complementary one-sided alternative hypothesis or the two-sided alternative hypothesis, $\sigma_3^2 \neq \sigma^2$, are considered in similar ways.

3.3 Variance Homogeneity Test, $r = 3$: The distributions of T in (9) and of S in (8) may be obtained when $r = 3$ under both H_{02} in (7) and the alternative hypothesis, $H_{a2}: \sigma_j^2 \neq \sigma_{j'}^2$, some $j \neq j'$, $j, j' = 1, 2, 3$. Indeed, under H_{a2}, the distribution of T is

$$f(T; \lambda) = \tfrac{1}{2}(n - 2)\lambda^{\frac{1}{2}(n-1)}T^{\frac{1}{2}(n-4)}G[\tfrac{1}{2}(n - 1), n/2, 1, (1 - T)(1 - \lambda)], \qquad (18)$$

where $T > 0$,

$$\lambda = 3 \sum_{j<j'} \sigma_j^2\sigma_{j'}^2/(\sum_j \sigma_j^2)^2, \qquad (19)$$

and G is the usual hypergeometric function. When H_{02} is true, $\lambda = 1$ and

$$f(T; 1) = \tfrac{1}{2}(n - 2)T^{\frac{1}{2}(n-4)}, \; T > 0, \qquad (20)$$

a very simple density. Furthermore, it follows at once that the density of $(n - 2)S/(n - 1)$ is the central chi-square distribution with 2 degrees of freedom, a result differing from the approximate one of Russell and Bradley (1958) only by the factor $(n - 2)/(n - 1)$.

The test of the hypothesis H_{02} may be based on $(n - 2)S/(n - 1)$ and the chi-square distribution, with large values of the statistic significant. The test may also be done simply in terms of T, with small values of T significant. Indeed, under H_{02} with $r = 3$, $P(T < \alpha^{2/(n-2)}) = \alpha$. The power of the α-level test is

$$P(T < \alpha^{2/(n-2)}|\lambda) = \int_0^{\alpha^{2/(n-2)}} f(T, \lambda)dT$$

for specified λ with $f(T, \lambda)$ given in (18).

3.4 Other Tests, $r = 3$: The joint densities (15) and (16) are relatively simple when $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$, $\sigma^2 = 1$ without loss of generality. Then we have

$$f(S_1, S_2, S_3) = \frac{3^{(n-1)/2}}{2^{n-1}\pi\Gamma(n-2)} [(\sum_j S_j)^2 - 2\sum_j S_j^2]^{(n-4)/2} e^{-\sum_j S_j/2}, \tag{21}$$

where $0 < S_j < 2\sum_{j'} S_{j'}/3$, $j = 1, 2, 3$, and $(\sum_j S_j)^2 - 2\sum_j S_j^2 > 0$,

and

$$f(Q_1,Q_2,Q_3) = \frac{(n-1)^{n-1}}{4\pi 3^{(n-1)/2}\Gamma(n-2)} (Q_1Q_2 + Q_1Q_3 + Q_2Q_3)^{(n-4)/2} e^{-(n-1)\sum_j Q_j/3}, \tag{22}$$

where $\sum_j Q_j > 0$ and $Q_1Q_2 + Q_1Q_3 + Q_2Q_3 > 0$. We have considered use of (21) to investigate the distribution of Shukla's statistic T^* for a test of variance homogeneity. The distribution of T^* is very intractable even when $r = 3$ and its use cannot be recommended in view of the simple results obtained above for T when $r = 3$. Also, use of (21) to rederive the distribution of Johnson's statistic, $S_j/S_{j'}$, $j \neq j'$, leads to no new insights or special simplifications.

3.5 A Joint Distribution, $r = 4$: The joint distribution of the elements of $\underset{\sim}{V}$ is given in (13), the number of variates on which $S_1, \ldots, S_r$ depend being $\frac{1}{2}r(r-1)$. We have been able to proceed to obtain the joint distribution of $2r - 3$ variates on which $S_1, \ldots, S_r$ depend, see Brindley (1982), but the joint distribution of $S_1, \ldots, S_r$ must be very complicated for general r and n.

The very special case with r = 4 and n = 4 has been completed. Then

$$f(S_1, S_2, S_3, S_4) = 2^{-1/2}\,\pi^{-3/2} \min_{1\le j\le 4}\{\sqrt{S_j}\}\, e^{-1/2 \sum_j S_j}$$

or

$$f(S_1, S_2, S_3, S_4) = (2\pi)^{-3/2}\Big(\sum_j \sqrt{S_j} - 2 \max_{1\le j\le 4}\{\sqrt{S_j}\}\Big)\, e^{-1/2 \sum_j S_j}$$

as $\sum_j S_j^2 - 2 \sum_{j<j'} S_j S_{j'}$ is less than $-8(\prod_j S_j)^{1/2}$ or between $-8(\prod_j S_j)^{1/2}$ and $8(\prod_j S_j)^{1/2}$ respectively, $S_j > 0$, $j = 1, \ldots, 4$, and $f(S_1, S_2, S_3, S_4)$ is zero otherwise.

It appears that the case with r = 3, an important case, may be the only situation for which simple distributions for test statistics may be found.

4. THE USE OF CHARACTERISTIC FUNCTIONS

Ellenberg (1977) used the method of characteristic functions to obtain the joint distribution of a subset of k of $S_1, \ldots, S_r$, $k \le r$, under the assumption that each $\sigma_j^2 = 1$. We summarize for k = r since coefficients in the series that he gives to represent the distribution are difficult to determine.

Let $\phi(\underset{\sim}{t})$ represent the characteristic function of $S_1, \ldots, S_r$, where $\underset{\sim}{t} = (t_1, \ldots, t_r)$. From Ellenberg's formula (3.2) for $\phi(\underset{\sim}{t})$, after algebraic reduction,

$$\phi(\underset{\sim}{t}) = \prod_j (1 - 2it_j)^{-(n-1)/2}\Big\{\frac{1}{r}\sum_j (1 - 2it_j)^{-1}\Big\}^{-(n-1)/2} \qquad (23)$$

where $i = \sqrt{-1}$ in (23). Since $(1 - 2it)^{-k/2}$ is the characteristic function of a chi-square variate with k degrees of freedom, it is appropriate to expand $\phi(\underset{\sim}{t})$ in negative powers of the $(1 - 2it_j)$. The result is that

$$\phi(\underset{\sim}{t}) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{n-1}{2})}{k!\Gamma(\frac{n-1}{2})} \sum_{s=0}^{k} \binom{k}{s}\left(-\frac{1}{r}\right)^s \underset{\sum_\alpha j_\alpha = s}{\Sigma\ldots\Sigma} \frac{s!}{\prod_\alpha j_\alpha!} \prod_\alpha (1 - 2it_\alpha)^{-(j_\alpha + \frac{n-1}{2})}$$

and

$$f(S_1, \ldots, S_r) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{n-1}{2})}{k!\Gamma(\frac{n-1}{2})} \sum_{s=0}^{k} \binom{k}{s}\left(-\frac{1}{r}\right)^s \underset{\sum_\alpha j_\alpha = s}{\Sigma\ldots\Sigma} \frac{s!}{\prod_\alpha j_\alpha!}$$

$$\prod_\alpha \frac{S_\alpha^{\,j_\alpha + \frac{n-1}{2} - 1}}{2^{\,j_\alpha + \frac{n-1}{2}}\,\Gamma(j_\alpha + \frac{n-1}{2})} \exp(-\tfrac{1}{2}\textstyle\sum_j S_j), \tag{24}$$

where α has the range 1, ..., r and $S_j > 0$, j = 1, ..., r, in (24).

It is far from apparent that (21) and (24) are identical when r = 3. Note that $(\sum_j S_j)^2 - 2\sum_j S_j^2 > 0$ for (21) but this requirement does not accompany (24). Two verifications have been made.

The moment generating function, $M_{\underset{\sim}{Q}}(\underset{\sim}{\theta}) = E\{\exp[(n - 1)\sum_j \theta_j Q_j/3]\}$, may be found from (22). This is done through use of the transformation, $u = Q_1$, $v = Q_1 + Q_2$, $w = Q_3 + Q_1Q_2/(Q_1 + Q_2)$, with integration with respect to w, u, and v in turn. The result is that

$$M_{\underset{\sim}{Q}}(\underset{\sim}{\theta}) = \{1 - \tfrac{2}{3}\textstyle\sum_j \theta_j + \tfrac{2}{3}(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) + \tfrac{1}{3}(\theta_1^2 + \theta_2^2 + \theta_3^2)\}^{-(n-1)/2}.$$

Since $S_1 = (n - 1)(4Q_1 + Q_2 + Q_3)/3$, $S_2 = (n - 1)(Q_1 + 4Q_2 + Q_3)/3$ and $S_3 = (n - 1)(Q_1 + Q_2 + 4Q_3)/3$, the moment generating function of

S_1, S_2 and S_3 is

$$M_{\underset{\sim}{S}}(\underset{\sim}{t}) = E(\exp \sum_j t_j S_j) = \{1 - \frac{4}{3} \sum_j t_j + \frac{4}{3}(t_1 t_2 + t_1 t_3 + t_2 t_3)\}^{-(n-1)/2},$$

corresponding to the characteristic function (23) for $r = 3$ from which (24) was derived.

In the very special case with $r = 3$ and $n = 4$, we have been able to sum the series in (24) to obtain the special form of (22). The method used seems awkward but an easier way has not been found. Details are given by Brindley (1982).

5. SIMULATIONS

Even though we have been unsuccessful in obtaining the joint distribution of $Q_1, \ldots, Q_r$ in usable form for $r > 3$, a test of variance homogeneity is needed. The statistic T in (9) yielded a simple distribution when $r = 3$ and has intuitive appeal, particularly when written in the alternative form,

$$T = \left\{1 - \frac{(Q_1 - Q_2)^2 + (Q_1 - Q_3)^2 + (Q_2 - Q_3)^2}{2(Q_1 + Q_2 + Q_3)^2}\right\}.$$

We consider the statistic,

$$X_r = -(n - 1) \log T_r, \tag{25}$$

where

$$T_r = 2r \sum_{j<j'} Q_j Q_{j'} / (r - 1)(\sum_j Q_j)^2. \tag{26}$$

Note that $T_3 = T$ and that $T_r = 1$ in the event that all Q_j are equal. Choice of the multiplier (n - 1) in (25) was made after empirical investigation.

Let us suppose that, when $\sigma_1^2 = \ldots = \sigma_r^2$, x_r/a_r has the central chi-square distribution with ν_r degrees of freedom. If the experiment is simulated a sufficient number of times, a_r and ν_r may be evaluated.

Two series of simulations have been conducted, the first to determine a_r and ν_r and the second to confirm and to extend the approximate procedure proposed to larger values of r. The method referenced in the Statistical Analysis Systems (SAS) user's guide was used to generate the required normal observations; properties of this generator are discussed by Lewis, Goodman and Miller (1969). In each simulation, nr independent observations on a standard normal variate were produced, grouped appropriately into n rows and r columns, and the value of X_r computed. For each r and n, 10,000 simulations were employed to compute the first and second moments of X_r from which a_r and ν_r were estimated. The first series of simulations were done for r = 4, 5 and 6, n = 10, 15, 20, 30 and 50. Results are given in Table I. Included also in Table I for comparison, are simulation results for r = 3 and the statistic $(n - 2)S/(n - 1) = (n - 2)X_3/(n - 1)$, S in (6), since this statistic is now known to have the chi-square distribution with 2 degrees of freedom.

The results shown in Table I are very interesting. It is apparent that ν_r is very close to (r - 1). Closer examination suggests that a_r is very close to 2/(r - 1)(r - 2). Accordingly, we suggest use of the statistic,

$$Y_r = -\frac{(n - 1)(r - 1)(r - 2)}{2} \log T_r, \tag{27}$$

TABLE I

Simulation Results to Determine a_r and ν_r

r	n	a_r	ν_r
3	10	1.007	1.957
	15	1.039	1.922
	20	1.006	1.982
	30	0.954	2.075
	50	0.972	2.031
4	10	0.332	2.994
	15	0.330	3.027
	20	0.327	3.023
	30	0.341	2.921
	50	0.332	2.990
5	10	0.168	3.871
	15	0.164	3.924
	20	0.162	3.985
	30	0.163	3.993
	50	0.162	3.955
6	10	0.103	4.687
	15	0.099	4.903
	20	0.097	4.973
	30	0.098	4.946
	50	0.097	4.973

TABLE II

Simulated Significance Levels of Y_r Compared to Those of χ^2_{r-1}

r	n	Right-tail Significance Levels for χ^2_{r-1}				
		0.01	0.05	0.10	0.20	0.50
3	10	0.010	0.048	0.098	0.199	0.502
	20	0.009	0.048	0.098	0.196	0.504
	30	0.011	0.049	0.098	0.200	0.491
	50	0.009	0.047	0.094	0.197	0.491
4	10	0.012	0.049	0.096	0.198	0.501
	20	0.011	0.048	0.098	0.200	0.502
	30	0.008	0.043	0.093	0.187	0.486
	50	0.007	0.046	0.098	0.197	0.489
5	10	0.012	0.048	0.093	0.184	0.486
	20	0.010	0.043	0.088	0.181	0.482
	30	0.011	0.047	0.095	0.191	0.487
	50	0.009	0.045	0.096	0.191	0.491
6	10	0.012	0.045	0.085	0.177	0.470
	20	0.009	0.045	0.088	0.180	0.474
	30	0.010	0.044	0.090	0.183	0.477
	50	0.008	0.042	0.085	0.177	0.477
8	20	0.009	0.045	0.086	0.175	0.459
	30	0.008	0.042	0.087	0.174	0.471
10	20	0.008	0.040	0.082	0.170	0.466
	30	0.009	0.040	0.084	0.177	0.470
12	20	0.009	0.042	0.082	0.164	0.450
	30	0.010	0.039	0.081	0.166	0.451

for the variance homogeneity test, Y_r to be taken to have the central chi-square distribution with $(r - 1)$ degrees of freedom. Percentage points of the simulated distribution of Y_r are compared with those of chi square in Table II for $r = 3, \ldots, 6, 8, 10$, and 12 for values of n shown.

The chi-square approximation to the distribution of Y_r seems remarkably good. In general, the approximation is very slightly conservative except for $n = 10$ at the .01-level, but even here the agreement is very good. The approximate method proposed seems to effectively provide the desired test of variance homogeneity based on Grubbs' estimators for practical purposes.

6. BRIEF EXAMPLES

We examine the variance homogeneity test for two examples in the literature. The first example has $r = 3$ and the exact test of Section 3.3 may be used. The second example has $r = 4$ and the approximate method of Section 5 may be used. In both examples, the null hypothesis has H_0: $\sigma_j^2 = \sigma^2$, $j = 1, \ldots, r$, and the alternative hypothesis is H_a: $\sigma_j^2 = \sigma_{j'}^2$ for some $j \neq j'$.

Russell and Bradley (1958) provided data on alcohol yields for three fermentors for $n = 38$ days in a distillery. Day effects were judged to be important so Grubbs' estimators were used. They calculated $Q_1 = 0.001537$, $Q_2 = 0.001722$ and $Q_3 = 0.000041$. The test statistic T in (9) has the value 0.7659. From (20), $P(T \leq T_0) = T_0^{(n-2)/2}$ and the P-value for the experiment is $(0.7659)^{18} = 0.0082$, indicating significantly different variances for the fermentors at the 0.01-level of significance.

Graybill (1954) gave the yields of $r = 4$ varieties of wheat at $n = 13$ locations in Iowa. It was suspected that error variances differed by varieties and that there were location effects. Calculation yields $Q_1 = 875.40$, $Q_2 = -84.92$, $Q_3 = 451.47$ and $Q_4 = 109.32$. Now $T_4 = 0.6109$ in (26) and Y_4 in (27) has the value 17.74, highly significant when compared with significance levels of chi-square with 3 degrees of freedom. These data were considered also by Ellenberg (1977), Snee (1982), and others.

7. REMARKS

Small-sample distribution theory based on Grubbs' estimators appears to be very difficult in general, but surprisingly simple when $r = 3$. Statistical methods are most needed for smaller values of r. We have provided the necessary theory when $r = 3$ and a good approximate test for variance homogeneity when $r > 3$. Some further investigation of the approximate test for small values of n may be desirable.

One warning should be issued. Tests on variances seem to be more sensitive to departures from the assumptions of normality than tests on means. This may be the case also for tests based on Grubbs' estimators and some investigation of the effects of nonnormality is suggested.

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Consider a two-way classification with n rows and r columns and the usual model of analysis of variance except that the error components of the model may have heterogeneous variances σ_j^2, $j = 1, \ldots, r$, by columns. Grubbs provided unbiased estimators Q_j of σ_j^2 that depend on column sums of squared residuals S_j. When $r = 3$, the joint distributions of the S_j and the Q_j are given for the first time in closed form.

Two tests proposed by Russell and Bradley are examined when $r = 3$, one for variance homogeneity and the second for one possible disparate variance. A very simple distribution is found for the test statistic of the first test and its non-null

20. Abstract (Continued)

distribution is derived also. The distribution of the second test statistic was known to be the central variance-ratio distribution in the null case and now its ratio to a parameter of noncentrality is shown to have the same distribution in the non-null case.

When $r = 4$, $n = 4$, the joint distribution of the S_j is given also in closed form, but it is difficult to use. For $r > 3$, an approximate test of variance homogeneity is proposed, based on an extension of the Russell-Bradley statistic. Extensive simulation studies show that the distribution of the test statistic may be approximated very well by a chi-square distribution.

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